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DIFFERENCE DISTRIBUTIONS APPLICABLE TO CERTAIN HEALTH PHYSICS MEASUREMENTS

Alan L. Justus, RP-SVS, 4/1/2019 (an update of LA-UR-16-23487)

Abstract - This paper discusses calculational methods for the determination of the difference distributions associated with certain health physics measurements. These measurements include the check source response counts relative to an initial reference count, the Albatross (i.e., HPI model 2080B) neutron tube counts relative to the gamma tube counts, and those that involve the use of the automatic background subtraction feature within portable health physics instrumentation. This paper therefore provides a technical basis for the necessary source strength of a check source in order to meet daily limits, the gamma field limitations of the HPI2080 Albatross, as well as the consequences of automatic background subtraction. Examples are provided that illustrate the methods for a few specific measurements.

INTRODUCTION

Certain health physics measurement applications involve the comparison of essentially two seemingly identical measurement results. A common example of this particular measurement application is the comparison of a morning's scaler count of a reference source to an essentially identical count performed sometime previously that established the so-called reference reading. The reference reading could have been established by an initial count characterized by the same counting time used in the subsequent daily checks, or it could have involved a count time that is, for instance, ten times that used for each morning's daily check. An additional example involving the comparison of essentially two identical gamma background measurements, takes place within the so-called Albatross pulsed neutron remmeter (i.e., Health Physics Instruments, Inc. (HPI) model 2080B), involving the 'running' comparison of the Ag-wrapped neutron Geiger-Mueller (GM) tube counts relative to the Sn-wrapped gamma-compensation GM tube counts. In the examples above, each of the two measurement results is randomly distributed about the same Poisson mean count value. The quantity of interest is the difference of the two results, which will be randomly distributed about the value of zero counts.

A slightly different health physics measurement application involves the comparison of two different measurement results. The example of this particular measurement application is the utilization of the automatic background subtraction feature within portable health physics instrumentation. Each of the two measurement results (i.e., the initial background count and the subsequent sample count) is randomly distributed about its own (and different) Poisson mean count value. The quantity of interest is the difference of the two results, which will be randomly distributed about the expected difference value, i.e., mean difference of counts.

This paper discusses the calculational methods for the determination of both of these so-called difference distributions, i.e., both for the former case involving the same mean count and

the latter involving different mean counts. Examples are provided which highlight the various concepts.

NORMAL APPROXIMATIONS FOR THE SAME MEAN COUNT CASE

Regarding the first case in the Introduction above (i.e., those applications involving the comparison of two measurements of the same mean count), if that same mean count value is expected to be hundreds of counts or greater, then the actual applicable Poisson distribution can be adequately approximated by a normal distribution. The difference distribution, randomly distributed about the expected difference value of zero counts, is then the difference of two normal distributions. This appears calculationally much simpler than dealing with the difference of the two Poisson distributions, and can readily be represented with analytical formulae known as the normal difference distribution (Appendix A).

Several Excel-based trial calculations were performed for eqn (A-2a), which represents the random expected difference in two measurements with the same mean count. Because Poisson statistics apply, the variance is set equal to the (Poisson) mean count. The calculations investigated mean counts ranging from 100 counts to 100,000 counts. As a first approximation, it was discovered that a mean count in the neighborhood of 2000 counts was necessary to reliably keep the difference within about 200 counts (i.e., $\pm 10\%$), and a mean in the neighborhood of 500 counts to reliably stay within about 100 counts (i.e., $\pm 20\%$). The calculations were then extended to include eqns (A-2b and c), which represented two measurements with the same mean count but with reference count times ten times and five times longer than the daily count time, respectively. As expected, the previously observed probabilities of exceeding $\pm 10\%$ or $\pm 20\%$ were significantly reduced. These lower probabilities are preferred, and indicate that when possible one should always choose a longer count time for the initial reference reading.

The two difference distributions, i.e., for the same count time and for the reference count time ten times longer, are plotted in Fig. 1 for the 2000 mean count example. Additionally plotted in Fig. 1 is the standard normal distribution, i.e., eqn (A-3), for a mean and variance of 2000 counts, but with the plotted distribution simply shifted downward 2000 counts (i.e., from being symmetrically distributed about 2000 counts to instead about zero counts). The difference distribution for the same count time is wider than the standard normal distribution (i.e., that labeled with 'single count' within Fig. 1) by the factor $\sqrt{2}$. The improvement with the longer reference count time is quite obvious (i.e., that labeled with 'ref. $\times 10$ ' within Fig. 1). In fact, that distribution's spread is quite similar to that expected in each daily gross count (i.e., the 'single count' curve). In other words, the variance in the daily difference would be dominated by the variance in that daily gross count rather than the variance in the initial determination of the reference value. The reader will recognize the similarity to the reason behind determining

background count rates using background count times that are ten times longer than the sample count times.

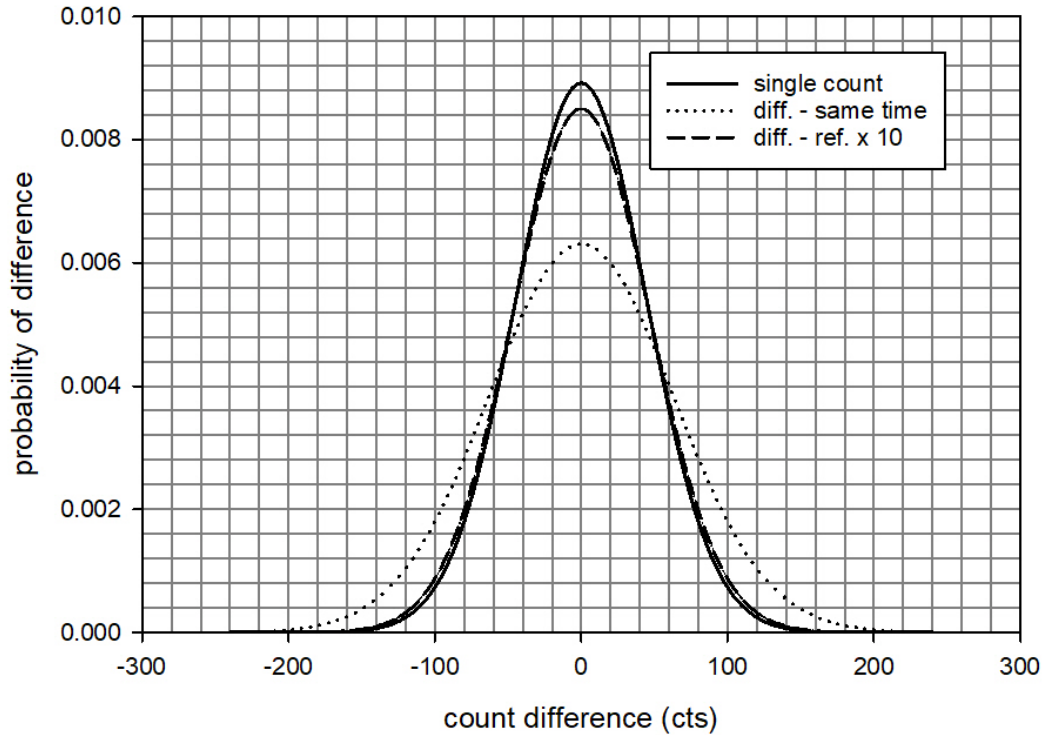


Figure 1. Widths of the normal difference distributions versus a standard normal distribution for a mean of 2000 counts (see text).

Mention was made above regarding ‘to reliably keep the difference within’ or ‘to reliably stay within’ such and such an amount. In the health physics measurements field, the magnitude of the amount can readily be surmised to be $\pm 10\%$ or 20% , but the magnitude of reliability requires further discussion. If the comparison is only performed weekly, then only about 50 comparisons are performed annually. If it is acceptable to deal with (i.e., recount) one ‘nuisance’ result per year, then the two-tailed nuisance probability is 2% and the magnitude of reliability is 98%. However, if the comparison is performed daily, then about 250 comparisons are performed annually. If it is again acceptable to have one ‘nuisance’ result per year, then the two-tailed nuisance probability is 0.4% and the magnitude of reliability is 99.6%. If the nuisance levels want to be made essentially negligible, then the nuisance rate can be further reduced by an order of magnitude to only once in 10-years. These possibilities are summarized in Table 1. For various reasons that will be explained later, the level of reliability preferred throughout the rest of this Section will be based on the negligible nuisance rate involving daily counts, i.e., 0.04%.

Table 1. Potentially-acceptable nuisance ‘alarm’ rates.

Tests per year	Acceptable nuisance ‘alarm’ rate	Two-tailed test nuisance probability	Equivalent one-tailed test probability
50 (i.e., one per week)	1 per year	0.02	0.01
	0.1 per year	0.002	0.001
250 (i.e., one per day)	1 per year	0.004	0.002
	0.1 per year	0.0004 (i.e., 0.04%)	0.0002

The exact minimum count necessary in a daily count of a reference source can now readily be determined in order to reliably meet established limits of $\pm 10\%$ or $\pm 20\%$. What will be presented here is based upon the requirement to yield a nuisance alarm rate of only 0.04% during a presumed daily counting interval. As described within Appendix A, the built-in Excel function NORM.INV was used in conjunction with eqns (A-2 and 4) with a one-sided cumulative probability of 0.9998. The results are presented in Table 2a. Note that any background counts are presumed to have negligible impact, either because background is essentially absent or absolutely stable. The numeric values are therefore in practice applicable only to alpha, neutron, and shielded and guarded beta measuring instrumentation with negligible background count rates.

Table 2a. Minimum gross counts necessary to meet established limits of $\pm 10\%$ and $\pm 20\%$ for a nuisance alarm rate of 0.04% (i.e., 1 in 2500). Note that background counts are presumed negligible.

	Paired-blank (x1) case	Reference x5 case	Reference x10 case
$\pm 10\%$	2505	1505	1378
$\pm 20\%$	627	376	345

Example 1. An alpha reference source needs to be selected for daily counts on the Ludlum model 43-10 alpha channel. The daily count time used will be 2-minutes. The reference reading, established initially for the entire year’s issuance of the instrument, will use a count time of 10-minutes. Established limits of $\pm 10\%$ are to be met. Referring to Table 2a, it can be seen that a minimum count rate of about 753 cpm (i.e., 1505 counts/2-minutes) is necessary. With a presumed counting efficiency of 0.39 cps Bq⁻¹, a minimum alpha activity of about 1930 dpm (33 Bq) is necessary. The probability of a (nuisance) statistical outlier is then negligible. If the result is beyond $\pm 10\%$, it probably isn’t statistics and, following a recount, an actual problem might need to be diagnosed.

When the background count rate is not negligible, the background count contribution to an increased variance must be addressed (Appendix A). Eqn (A-5) was used to account for the increased standard deviation for a background count of 60 counts (typifying pancake GMs and energy-compensated GMs), a background count of 600 counts (typifying the beta channels of P-10 gas proportional counter (PC) and dual-scintillator detectors), and a background count of 6000 counts (typifying NaI(Tl)-based gamma scintillation detectors). The resulting minimum net counts are presented in Tables 2b-d, respectively. For intermediate values of background count, the results are additionally presented in Figures 2a-b. Note that the background count is the product of the background count rate and the sampling time.

Table 2b. Minimum net counts necessary to meet established limits of $\pm 10\%$ and $\pm 20\%$ for a nuisance alarm rate of 0.04% (i.e., 1 in 2500) and with a background count of 60 counts.

	Paired-blank (x1) case	Reference x5 case	Reference x10 case
$\pm 10\%$	2620	1615	1490
$\pm 20\%$	730	472	439

Table 2c. Minimum net counts necessary to meet established limits of $\pm 10\%$ and $\pm 20\%$ for a nuisance alarm rate of 0.04% (i.e., 1 in 2500) and with a background count of 600 counts.

	Paired-blank (x1) case	Reference x5 case	Reference x10 case
$\pm 10\%$	3395	2290	2150
$\pm 20\%$	1235	886	838

Table 2d. Minimum net counts necessary to meet established limits of $\pm 10\%$ and $\pm 20\%$ for a nuisance alarm rate of 0.04% (i.e., 1 in 2500) and with a background count of 6000 counts.

	Paired-blank (x1) case	Reference x5 case	Reference x10 case
$\pm 10\%$	6880	5065	4815
$\pm 20\%$	3073	2320	2213

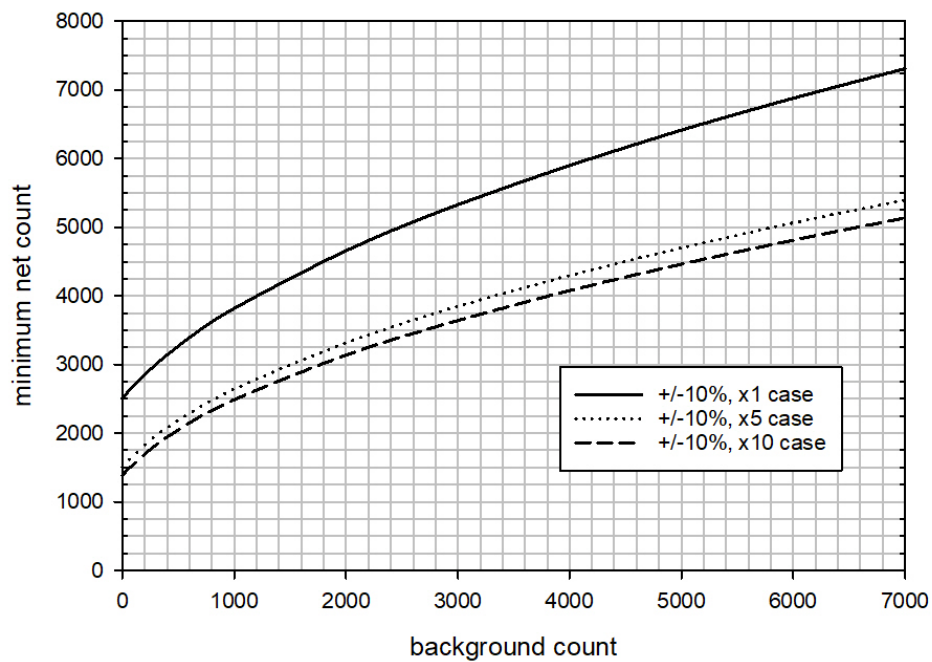


Figure 2a. Minimum net counts necessary to meet established limits of $\pm 10\%$ for a nuisance alarm rate of 0.04% (i.e., 1 in 2500) for background counts up to 7000 counts.

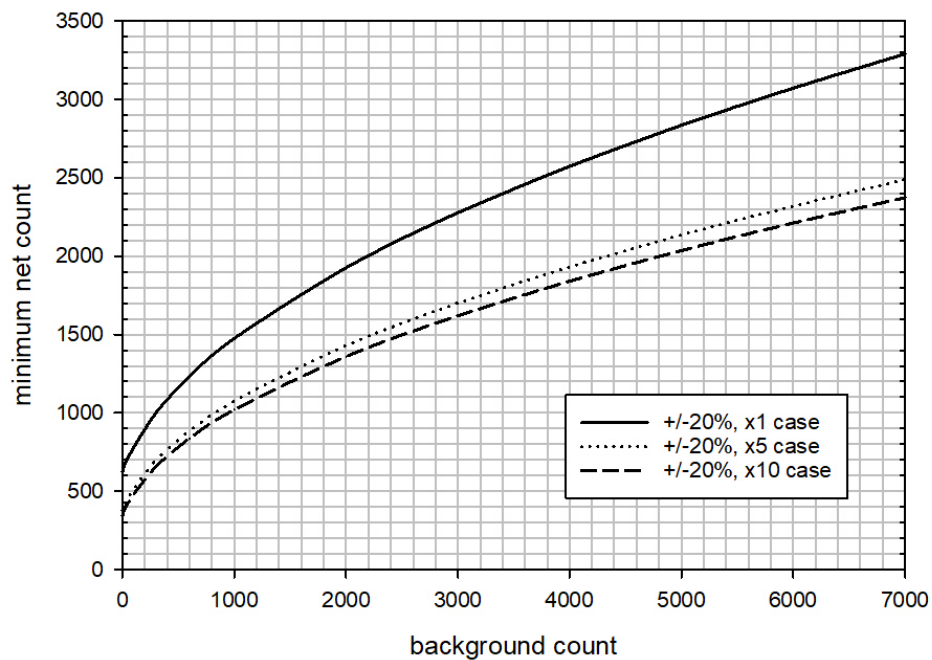


Figure 2b. Minimum net counts necessary to meet established limits of $\pm 20\%$ for a nuisance alarm rate of 0.04% (i.e., 1 in 2500) for background counts up to 7000 counts.

AC-powered laboratory-grade alpha/beta counting systems should generally be able to meet a $\pm 10\%$ criterion. Using Tables 2a and c, the minimum reference source activity necessary for several representative manufacturer's models and typical scaler count times is presented in Table 3. The alpha efficiencies used for the Eberline model SAC-4, Ludlum model 43-10-1, automatic planchet counter models, and the Berthold models were 0.36, 0.39, 0.38, and 0.34 cps Bq⁻¹, respectively. The beta efficiencies used for the Ludlum model 43-10-1, automatic planchet counter models, and the Berthold models were 0.41, 0.45, and 0.50 cps Bq⁻¹, respectively. All backgrounds were negligible with the exception of the beta channel of the Ludlum 43-10-1.

Table 3. *Minimum* laboratory instrument check source strengths necessary to reliably meet established limits of $\pm 10\%$. The reference count is presumed to be 5 times longer than the daily count time. (The level of reliability is 1 nuisance 'alarm' every 2500 measurements.)

Model	Count Time (min)	dpm (Bq) alpha	dpm (Bq) beta
Eberline SAC-4 (manual planchet counter, ZnS scintillator based)	1	4180(70)	NA
Ludlum M43-10-1 (manual planchet counter, dual scintillator based)	2	1930(32)	2800(47)
Mirion/Protean (automatic planchet counter, P-10 gas PC)	2	1980(33)	1670(28)
Berthold 770/790 (wide-area, P-10 gas PC)	5	885(15)	600(10)

Portable health physics instruments should generally be able to meet a $\pm 20\%$ criterion. The minimum reference source activity, or source strength, necessary is presented in Table 4 for some common detectors in use at LANL for various ratemeter response times as well as for a scaler count time of 60-seconds. The response time is considered here to be 2.2τ , where τ is the characteristic time constant of the ratemeter response, and 2τ is the effective sampling time (Evans 1955). The background count rates used for the Eberline model HP380AB-alpha, HP380AB-beta, various pancake GMs, Eberline model HP-270 energy-compensated GM, Eberline model NRD-1 neutron remmeter, and Ludlum 44-10 2×2 NaI were 8, 450, 70, 20, 0, and 7000 cpm, respectively. From the effective sampling time and the given background count rate, the background count is determined, and Figs. 2a-b then used to determine the minimum net count. The efficiencies used for the Eberline model HP380AB-alpha, HP380AB-beta, various pancake GMs, Eberline model HP-270 energy-compensated GM, and Eberline model NRD-1

neutron remmeter were 0.2 cps Bq⁻¹, 0.3 cps Bq⁻¹, 0.34 cps Bq⁻¹, 20 cps per mR/h, and 2.33 cps per mrem/h, respectively.

Table 4. *Minimum* portable instrument check source readings necessary to reliably meet established limits of $\pm 20\%$. The paired-blank case is assumed. (The level of reliability is 1 nuisance ‘alarm’ every 2500 measurements.)

Mode	Response Time (s)	HP380AB-alpha dpm(Bq)	HP380AB-beta dpm(Bq)	Pancake-beta dpm(Bq)	HP-270-mR/h	NRD-mrem/h	44-10 cpm gamma
Ratemeter-							
Fast	3	69k(1150)	48k(810)	41k(680)	12	99	22000
Medium	10	21k(345)	16k(270)	13k(210)	3.4	30	9900
Standard	22	9.4k(157)	8.2k(136)	5.9k(98)	1.6	14	6100
Slow	30	6.9k(115)	6.5k(110)	4.4k(74)	1.1	10	5100
Scaler-	60	3.1k(52)	3.7k(62)	2.2k(36)	0.6	4.5	3300

Example 2. A neutron reference field reading needs to be selected for daily readings with the Eberline model NRD-1 coupled to the Thermo model RadEye PX ratemeter. The effective (i.e., 2τ) count time used will be 20 seconds (i.e., the ‘standard’ response time). Established limits of $\pm 20\%$ are to be met. Referring to Table 4, it can be seen that a minimum reading of about 14 mrem/h is necessary. The probability of a statistical outlier is quite small. If the result is beyond $\pm 20\%$, it probably isn’t statistics, and an actual problem might need to be diagnosed. Although source strengths should minimally yield 14 mrem/h, larger values are acceptable and encouraged. It should be realized that with larger values, the expected variations in response test readings will be tighter, and actual drifts due to a problem might go undetected for some time.

Example 3. For hand and shoe monitors (HFMs) and personnel contamination monitors (PCMs), the count time at LANL is 16 s, and typical alpha and beta efficiencies are 0.15 and 0.2 cps Bq⁻¹, respectively. With typical background count rates of 0.2 and 20 cps in the alpha and beta channels, respectively, the background counts are 3.2 and 320 counts, respectively. From Table 2a and Fig. 2b, the required alpha and beta check source count rates are therefore 39.2 cps (i.e., 627/16) and 62.5 cps (i.e., 1000/16), respectively, requiring minimum alpha and beta activities of 15,680 dpm (261Bq) and 18,750 dpm (313 Bq), respectively, to reliably stay within $\pm 20\%$.

EXACT POISSON TREATMENT FOR THE SAME MEAN COUNT CASE

It has so far been assumed that, as long as the mean count was hundreds of counts or more, that the applicable Poisson distribution can be adequately approximated by a normal distribution. The difference distribution, randomly distributed about the value of zero counts, was then the difference of two normal distributions. However, when the mean count is

significantly lower, the actual difference of two Poisson distributions must instead be determined. The calculational methods are detailed within Appendix B.

Several Excel-based trial calculations were performed using eqn (B-1), which represented the expected random difference in two measurements with the same Poisson mean count. The calculations investigated Poisson mean counts initially ranging from 0.5 mean counts to 8 mean counts per counting interval. The Poisson difference distribution for the Poisson mean count equal to 1 is plotted in Fig. 3. Note that only one side of the symmetric distribution is shown. Additionally plotted in Fig. 3 is the normal difference distribution approximation, i.e., eqn (A-2a) for a mean and variance equal to 1 count. Differences between the two probability distributions are apparent at probabilities that are about one order of magnitude beneath the most probable (i.e., that at zero). As the Poisson mean count increased, the departure of the normal difference distribution approximation from the exact Poisson treatment decreased. The Poisson difference distribution for an 8 count mean is plotted in Fig. 4, along with the normal difference distribution approximation, i.e., eqn (A-2a) for a mean and variance equal to 8 counts. At 8 counts, the difference between the two distributions is negligible out to probabilities that are about two orders of magnitude beneath the most probable (i.e., that at zero).

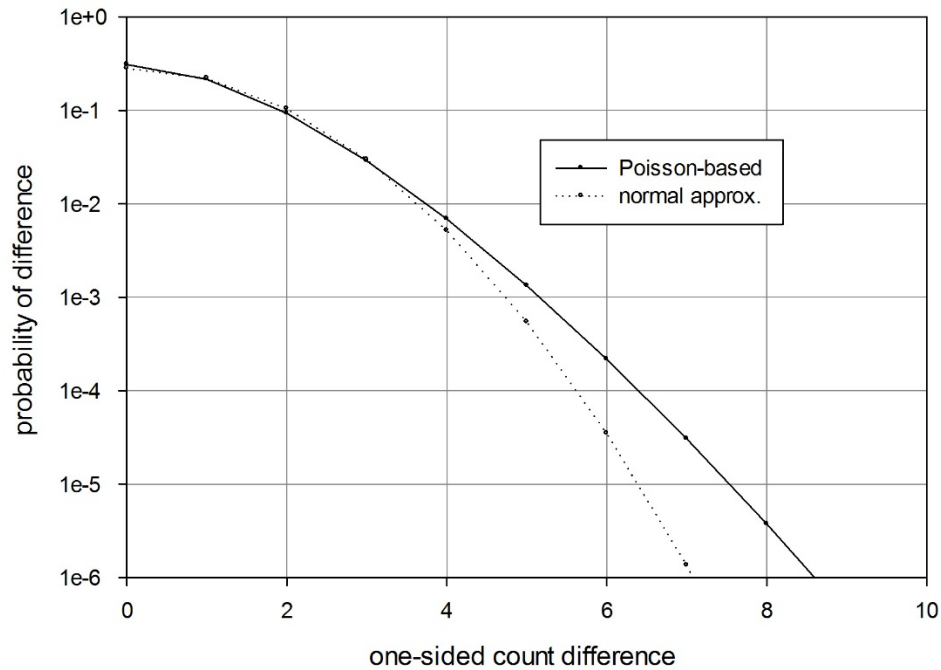


Figure 3. One side of the difference distribution for Poisson means of 1 count.

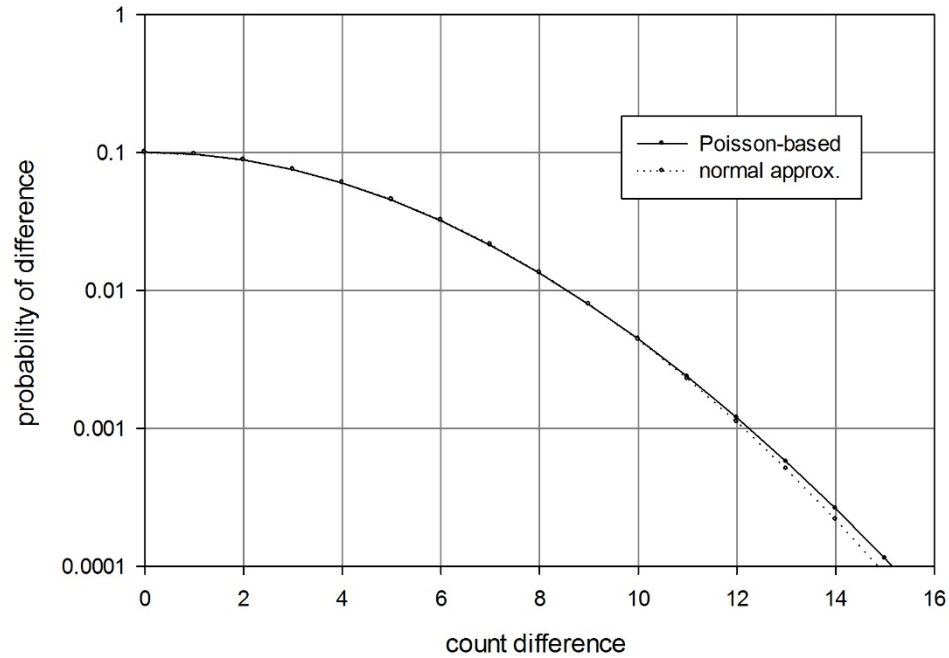


Figure 4. One side of the difference distribution for Poisson means of 8 counts.

Within the so-called Albatross pulsed neutron remmeter (i.e., specifically the HPI model 2080B pulse neutron survey meter), a comparison is made of the neutron-plus-gamma-sensitive Ag-wrapped GM tube counts relative to the gamma-only-sensitive Sn-wrapped gamma-compensation GM tube counts. The counting interval is adjustable in 16-second increments from 16-s (AV=1) up to 512-s (AV=32). The GM tubes have an identical gamma sensitivity of 6 cps per mR/h. Any net positive neutron tube count difference is converted to a displayed neutron dose reading through the neutron calibration factor of 0.2 cps per mrem/h. (Note: at the present time, the display actually only reads in mrem/h.)

For a counting interval equal to just one averaging time (AV=1), one can be interested as to what the neutron readings could potentially reach up to in gamma fields ranging from about 5 to 80 μ R/h. As can be seen in Table 5a, Poisson mean counts ranging from about 0.5 counts up to about 8 counts per 16-s interval are considered. Eqn (B-4) was utilized within Excel to calculate each difference distribution. Column 4 lists the practical maximum count difference that can be reached, obtained from the cumulative distribution function (cdf) of each calculated probability distribution. The resulting neutron tube net count rate, temporarily reached $\leq 1\%$ of the time, is listed in column 5. Through the neutron dose rate calibration factor of 0.2 cps per mrem/h, the resulting temporarily displayed reading is given in the last column. Note that, due

to ambient gamma fields, temporary neutron readings are possible that are potentially about two orders of magnitude greater than the gamma fields.

The effective way to reduce this influence is to increase the counting interval from 16-s (AV=1) up to 128-s (AV=8) or more, since the relative error in the Poisson difference distribution is observed to still exhibit the $1/\sqrt{\text{count}}$ dependence typical of the individual Poisson distributions themselves. This is demonstrated in Table 5b.

Table 5a. Effect of gamma fields on the 2080B neutron reading for AV=1 (i.e., 16-s).

Ambient gamma (μR/h)	Mean count rate (cps)	Poisson mean (counts/16-s)	Max. reach ($\leq 1\%$) (counts/16-s)	Max. reach (cps)	Temp. reading (mrem/h)
5	0.03	0.48	2	0.13	0.63
10	0.06	0.96	3	0.19	0.94
20	0.12	1.92	5	0.31	1.6
40	0.24	3.84	7	0.44	2.2
80	0.48	7.68	9	0.56	2.8

Table 5b. Effect of gamma fields on the 2080B neutron reading for AV=8 (i.e., 128-s).

Ambient gamma (μR/h)	Mean count rate (cps)	Poisson mean (counts/128-s)	Max. reach ($\leq 1\%$) (counts/128-s)	Max. reach (cps)	Temp. reading (mrem/h)
5	0.03	3.84	7	0.05	0.27
10	0.06	7.68	9	0.07	0.35
20	0.12	15.4	13	0.10	0.51
40	0.24	30.7	19	0.15	0.74
80	0.48	61.4	27	0.21	1.05

Note that if actual neutron fields are also present, then these gamma-induced fluctuations will simply be superimposed upon the actual neutron readings (rather than the previously presumed 0 mrem/h neutron field). So far, the ambient gamma fields were assumed of course to be steady-state gamma fields. In pulsed fields, the influence of narrow accelerator pulses of photon radiation are effectively cancelled due to the occurrence of a simultaneous count in both the neutron and gamma GM tubes. The additional dead-time thus introduced is normally quite trivial to the live time of the neutron measurement.

NORMAL APPROXIMATIONS FOR DIFFERENT MEAN COUNTS

Regarding the latter case in the Introduction, i.e., those applications involving the comparison of two different mean counts, if those mean count values are expected to be

hundreds of counts or greater, then once again each applicable Poisson distribution can be adequately approximated by a normal distribution. This is typically the case, for instance, for the beta counting channels of COTS (commercially-available off the shelf) portable health physics instrumentation. The difference distribution, randomly distributed about the mean difference of counts, is then the difference of two normal distributions. As seen previously, this is calculational quite simple, readily represented by the normal difference distribution (detailed within Appendix A).

Several Excel-based calculations were performed for eqn (A-6), which represents two measurements (i.e., a background count and a gross sample count) with the same count time. Because Poisson statistics apply, each variance was set equal to its Poisson mean count. Since a portable health physics instrument's beta counting channel typically exhibits background count rates of 300 to 500 cpm, the background mean count for a 60-s scaler would range from 300 to 500 counts. The beta activities of interest will be assumed to be 0 dpm (0 Bq), 200 dpm (3.33 Bq), and 1000 dpm (16.66 Bq). (The last two values taken directly from the DOE unrestricted release tables.) Since the beta counting channel of the portable instrument typically exhibits efficiencies of approximately 0.3 cps Bq^{-1} , net mean count rates of interest are therefore 0 cpm (i.e., representing no net activity in the sample), 60 cpm (i.e., representing $3.33 \text{ Bq} \times 0.3 \text{ cps Bq}^{-1}$), and 300 cpm (i.e., representing $16.66 \text{ Bq} \times 0.3 \text{ cps Bq}^{-1}$). Net mean counts of interest for a 60-s scaler are therefore 0, 60, and 300 counts. Gross mean counts are simply the sum of the appropriate background count and net count.

The difference distribution for the background of 300 counts and a net of 0 counts is plotted in Fig. 5a (left side). This represents the possible measurement values with background subtraction activated within the instrument. Although this difference distribution is actually symmetric about the value of 0 counts, COTS health physics instrumentation (such as the Thermo E-600 and RadEye) do not display negative numbers. Hence, a mean value of 0 counts is quite difficult to confirm. The difference distribution of Fig. 5a effectively should show a zero probability for each count < 0 and a value of 0.5163 at 0 counts (i.e., $0.5 + 0.0163$). Additionally plotted in Fig. 5a (right side) is the standard normal distribution, i.e., eqn (A-3), for a mean and variance of 300 counts. This represents the possible measurement values without background subtraction activated (i.e., simple gross mode). Here, the mean value of 300 counts is quite easy to observe, even in ratemeter mode. Fig. 5b replots the distributions, but with the gross count distribution simply shifted downward by 300 counts, i.e., from being symmetrically distributed about 300 counts to instead about 0 counts (in order to compare relative shapes). Note that Fig. 5b is just a one-sided form of Fig. 1. As also seen in the first section, the difference distribution for the net count is found to be wider than the standard normal distribution (i.e., that labeled 'gross bkgd of 300' within Fig. 5) by the factor of $\sqrt{2} = 1.41$, exactly that expected from classical statistics for the critical level (Currie 1968). In other words, the variance in the net count is doubled since it is the sum of the background variance and the gross count variance (which is the same as that for background for a net of 0 counts).

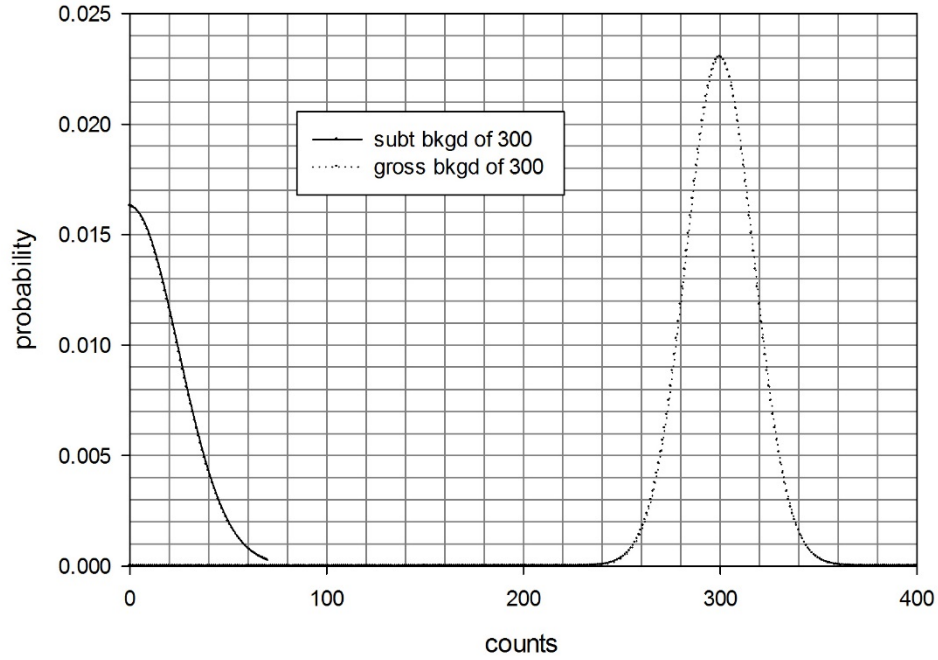


Figure 5a. Left side: the difference distribution for the background of 300 counts and a net of 0 counts (i.e., background subtraction mode). Right side: the expected possibilities in simple gross mode (with 300 counts).

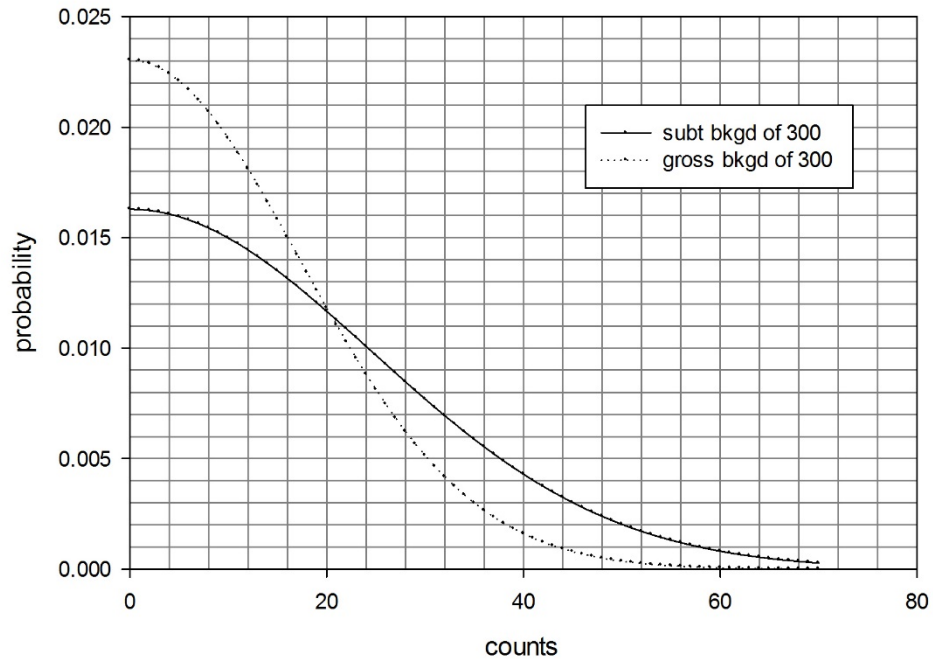


Figure 5b. A comparison of the relative widths of the two distributions plotted in Fig. 5a.

Next, the difference distribution for the background of 300 counts and a net of now 60 counts is plotted in Fig. 6a (left side). This represents the possible measurement values with background subtraction activated within the instrument. In this case, this difference distribution is symmetric about the value of 60 counts, which is now easier to confirm with health physics instrumentation. Additionally plotted in Fig. 6a (right side) is the standard normal distribution, i.e., eqn (A-3), for a mean and variance of 360 counts. This represents the possible measurement values without background subtraction activated (i.e., simple gross mode). Here, the mean value of 360 counts is also very easy to observe, even in ratemeter mode. Fig. 6b replots the distributions, but with the gross count distribution simply shifted downward by 300 counts (i.e., from being symmetrically distributed about 360 counts to instead about 60 counts). The difference distribution for the net count is found to be just slightly wider than the standard normal distribution (i.e., that labeled ‘gross bkgd of 300’ within Fig. 6) by the factor of $\sqrt{(660/360)} = \sqrt{1.83} = 1.35$, exactly that expected from classical statistics for the critical level (Currie 1968). The variance in the net count is the sum of the background variance (i.e., 300) and the gross count variance (i.e., 360).

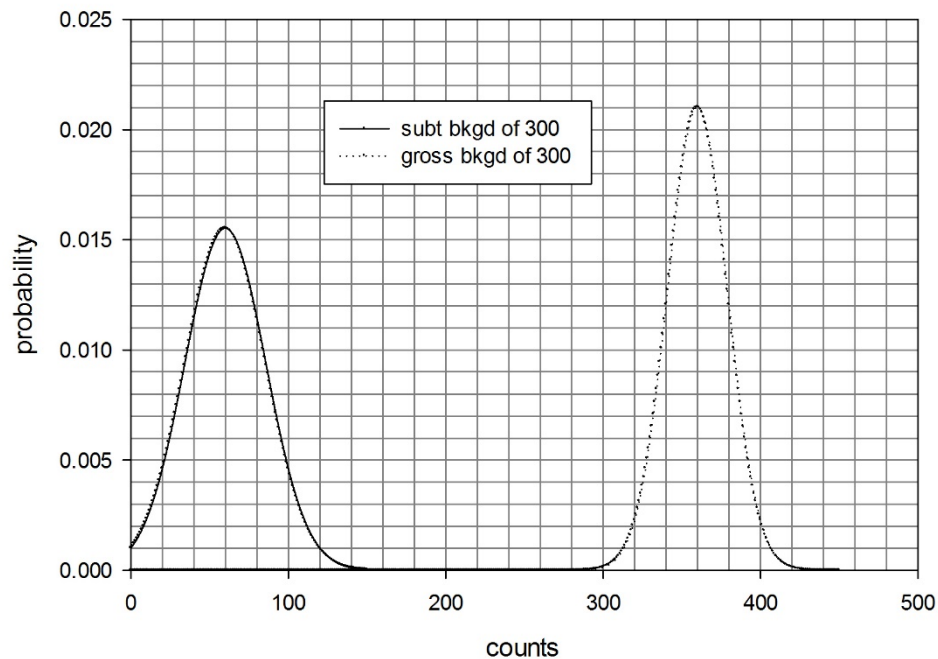


Figure 6a. Left side: the difference distribution for the background of 300 counts and a net of 60 counts (i.e., background subtraction mode). Right side: the expected possibilities in simple gross mode (with 360 counts).

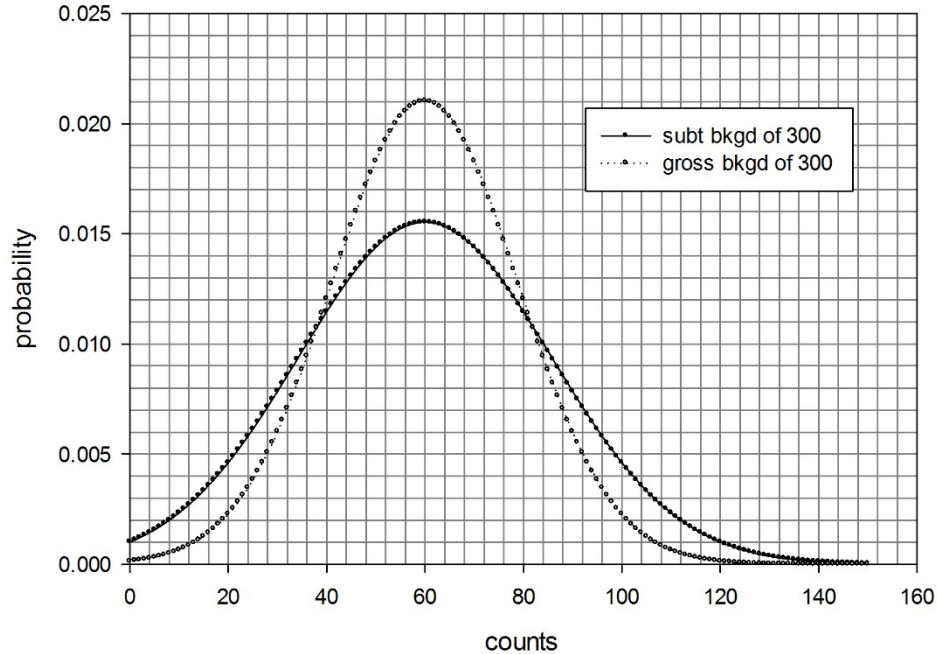


Figure 6b. A comparison of the relative widths of the two distributions plotted in Fig. 6a.

Since no new discoveries would be conveyed by presenting the very similar results involving either the background count of 500 counts or the net count value of 300 counts, such presentations will be forgone. It should simply be noted that (for most COTS health physics portable instruments utilizing automatic background subtraction), net count values near 0 counts will have the negative values of the difference distribution truncated to 0 counts, whereas significant net count values would be displayed correctly, but with a slightly larger associated variance than the gross count itself. In addition, in many of these COTS instruments, the stored background value can either be difficult to access or in a unit of measurement that is different than that on the instrument's display, e.g., 'cps' with a 'dpm' display.

EXACT POISSON TREATMENT FOR DIFFERENT MEAN COUNTS

It was assumed in the previous section that as long as the mean count was a relatively large count value, that the applicable Poisson distribution can be adequately approximated by a normal distribution. However, at the very low count rates associated with the alpha counting channels of portable health physics instrumentation, this approximation is not valid. Therefore, additional Excel-based calculations were performed, which represented the expected difference in two measurements (i.e., a background count and a gross sample count) with different Poisson mean counts (see Appendix C for the calculational details).

Several Excel-based calculations were performed for eqn (C-4), which represents two measurements (i.e., a background count and a gross sample count) with the same count time.

Since a portable health physics instrument's alpha counting channel typically exhibits background count rates of 1 to 7 cpm, the values considered for the background mean count for a 60-s scaler are 1, 4, and 7 counts. The alpha activities of interest will be assumed to be 0 dpm (0 Bq), 20 dpm (0.33 Bq), and 50 dpm (0.833 Bq). (The last two values are related to the DOE unrestricted release tables.) Since the alpha counting channel of the portable instrument typically exhibits efficiencies of approximately 0.2 cps Bq^{-1} , net mean count rates of interest are therefore 0 cpm (i.e., representing no net activity in the sample), 4 cpm (i.e., representing $0.33 \text{ Bq} \times 0.2 \text{ cps Bq}^{-1}$), and 10 cpm (i.e., representing $0.833 \text{ Bq} \times 0.2 \text{ cps Bq}^{-1}$). Net mean counts of interest for a 60-s scaler are therefore 0, 4, and 10 counts. Gross mean counts are simply the sum of the appropriate background count and net count.

The Poisson difference distributions for a net of 0 counts and for all 3 chosen background count values are plotted in Fig. 7a (left side). These represent the possible measurement values with background subtraction activated within the instrument. Although the difference distributions are actually symmetric about the value of 0 counts, COTS health physics instrumentation (such as the Thermo E-600 and RadEye) do not typically display negative numbers. Hence, the mean value of 0 counts is very difficult to confirm. Additionally plotted in Fig. 7a (toward the right side) are the standard Poisson distributions for means equal to the gross (i.e., background) count values. These represent the possible measurement values without background subtraction activated (i.e., simple gross mode). Fig. 7b replots the distributions, but with the gross count distributions simply shifted downward by the gross count values (in order to compare relative shapes). Note that Fig. 7a (left side) is essentially a two-sided form of Figs. 3 and 4. As also seen in the second section, the Poisson difference distributions for the net count are found to be wider than the standard Poisson distribution (i.e., those labeled 'gross bkgd of ...' within Fig. 7) by the factor of $\sqrt{2}$, which is exactly that expected from classical statistics for the critical level (Currie 1968). In other words, the variance in the net count is doubled since it is the sum of the background variance and the gross count variance (which is the same as the background for a net of 0 counts).

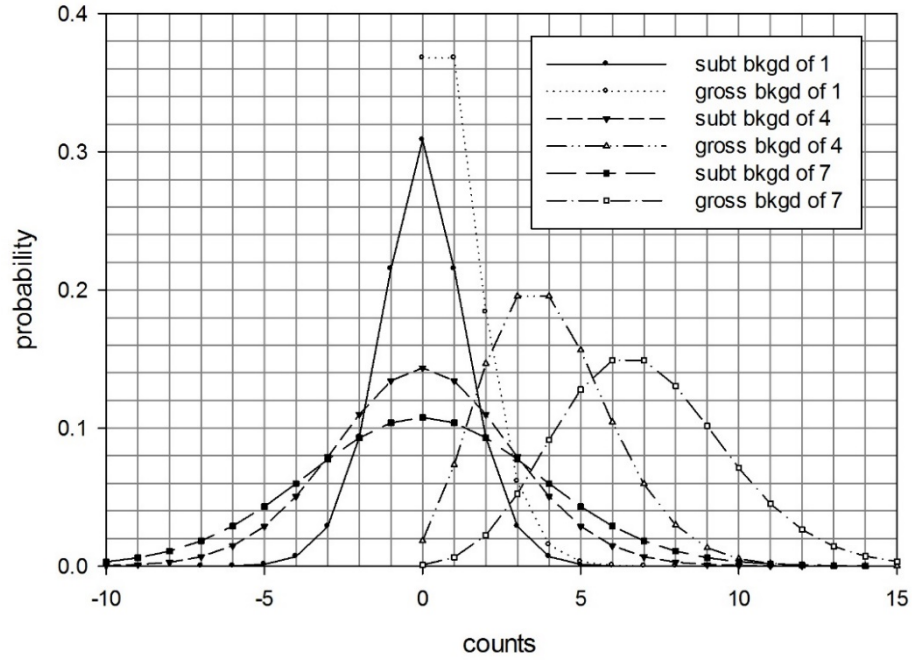


Figure 7a. Left side: difference distributions for a net of 0 counts and each selected background count (i.e., background subtraction mode). Right side: the expected possibilities in simple gross mode (with a net of 0 counts).

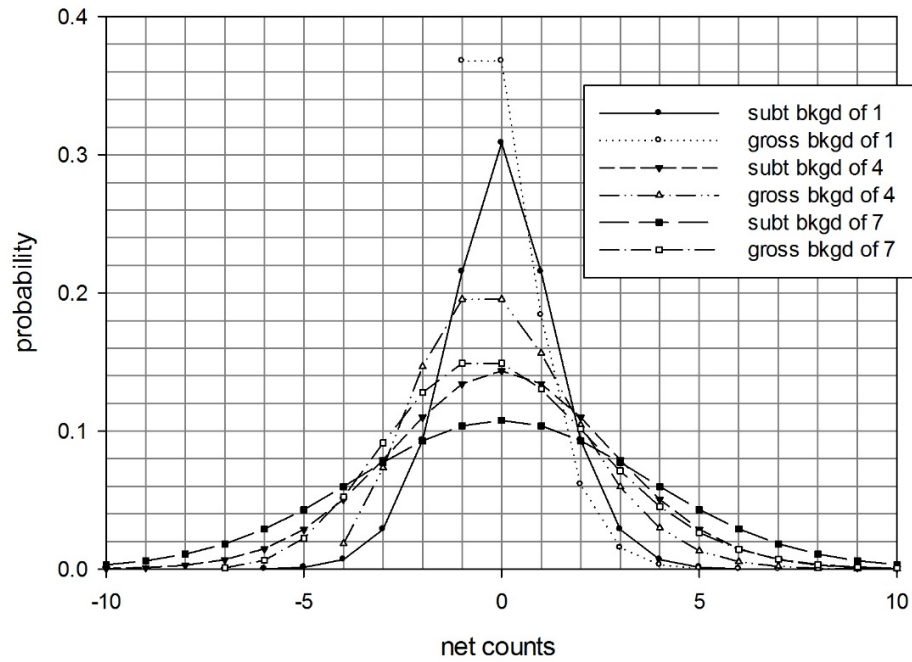


Figure 7b. A comparison of the relative widths of the distributions plotted in Fig. 7a.

Next, the Poisson difference distributions for a net of 4 counts and for all 3 chosen background count values are plotted in Fig. 8a (left side). These represent the possible measurement values with background subtraction activated within the instrument. Although the difference distributions are now fairly symmetric about the value of 4 counts, COTS health physics instrumentation still do not display the negative members of the distributions. Hence, the mean value of 4 counts is still somewhat difficult to confirm. Additionally plotted in Fig. 8a (right side) are the standard Poisson distributions for means equal to the gross count values (i.e., background + 4 counts). These represent the possible measurement values without background subtraction activated (i.e., simple gross mode). Fig. 8b replots the distributions, but with the gross count distributions simply shifted downward by each background count value. The Poisson difference distributions for the net count are once again found to be wider than the standard Poisson distributions (i.e., those labeled ‘gross bkgd of ...’ within Fig. 8). The variance in the net count is the sum of the background variance (i.e., 1, 4, or 7) and the gross count variance (i.e., 1+4, 4+4, or 7+4, respectively).

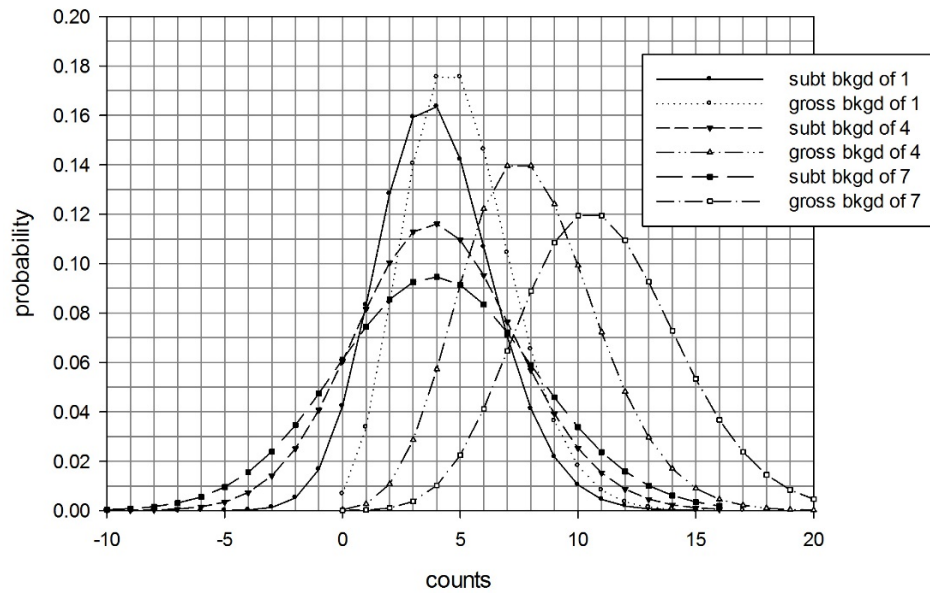


Figure 8a. Left side: difference distributions for a net of 4 counts and each selected background count (i.e., background subtraction mode). Right side: the expected possibilities in simple gross mode (with a net of 4 counts).

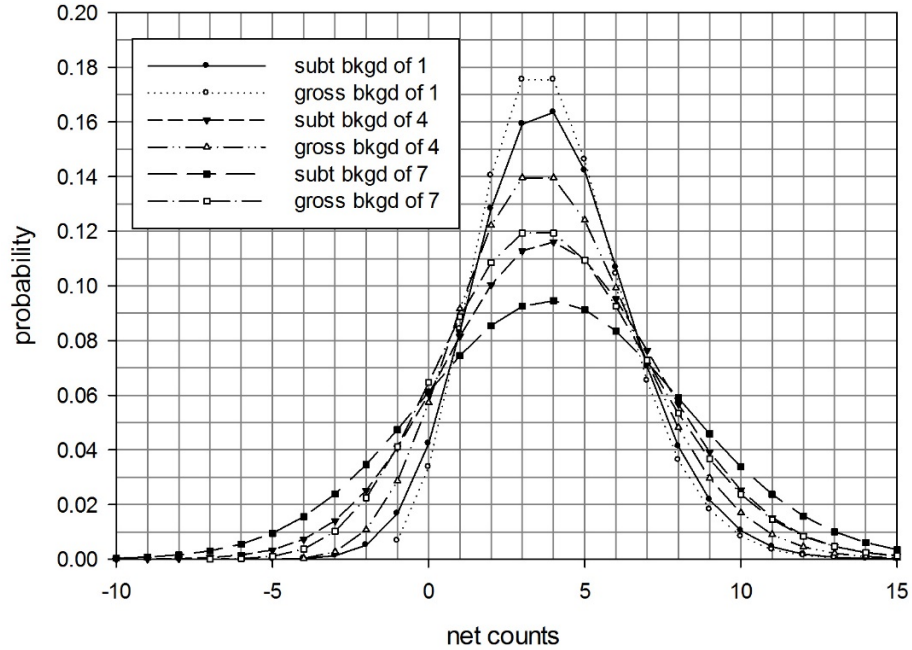


Figure 8b. A comparison of the relative widths of the distributions plotted in Fig. 8a.

Finally, the Poisson difference distributions for a net of 10 counts and for all 3 chosen background count values are plotted in Fig. 9a (left side). These represent the possible measurement values with background subtraction activated within the instrument. Although the difference distributions are now fairly symmetric about the value of 10 counts, health physics instrumentation still do not display the few negative members of the distributions. Yet, the mean value of 10 counts is now much easier to confirm than the previous smaller values. Additionally plotted in Fig. 9a (right side) are the standard Poisson distributions for means equal to the gross count values (i.e., background + 10 counts). These represent the possible measurement values without background subtraction activated (i.e., simple gross mode). Fig. 9b replots the distributions, but with the gross count distributions simply shifted downward by each background count value. The Poisson difference distributions for the net count are once again found to be wider than the standard Poisson distributions (i.e., those labeled ‘gross bkgd of ...’ within Fig. 9). The variance in the net count is the sum of the background variance (i.e., 1, 4, or 7) and the gross count variance (i.e., 1+10, 4+10, or 7+10, respectively).

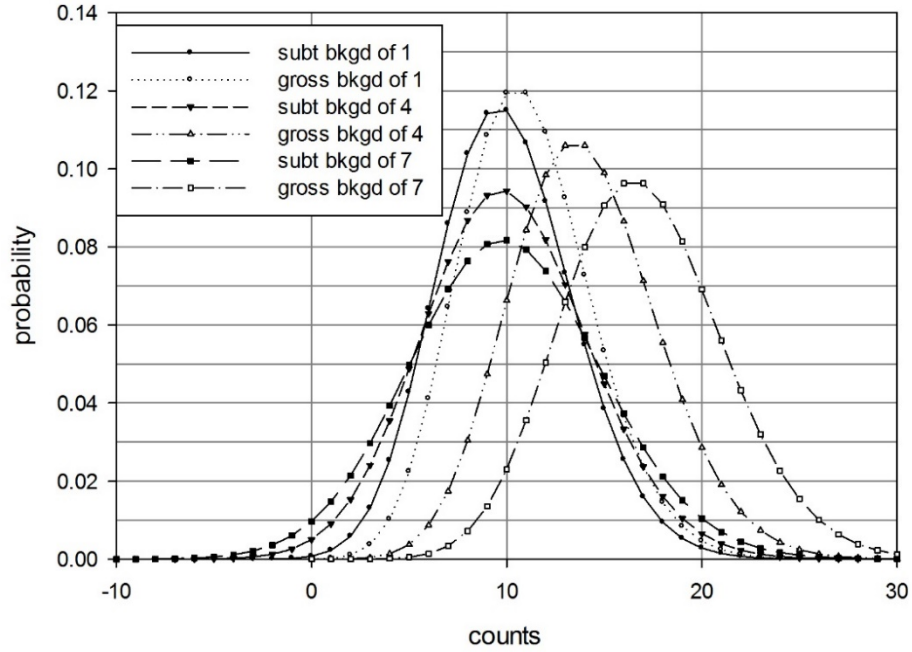


Figure 9a. Left side: difference distributions for a net of 10 counts and each selected background count (i.e., background subtraction mode). Right side: the expected possibilities in simple gross mode (with a net of 10 counts).

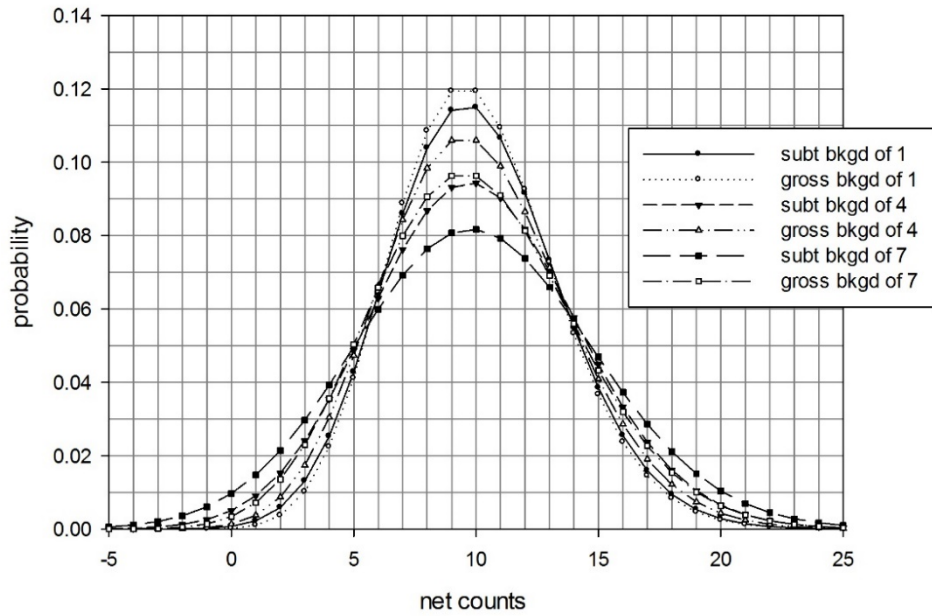


Figure 9b. A comparison of the relative widths of the distributions plotted in Fig. 9a.

As in the previous section (i.e., for the beta counting channels), it should again be noted that for most COTS health physics portable instruments utilizing automatic background subtraction, net alpha count values near 0 counts will always have the negative values of the difference distribution truncated to 0 counts, whereas significant net count values would be displayed correctly, but with a slightly larger associated variance than the gross count itself. In addition, in many of these COTS instruments, the stored alpha background value can either be difficult to access or in a unit of measurement that is different than that on the instrument's display, e.g., cps with a dpm display.

DISCUSSION

Difference distributions (the distribution formed from the random difference between two random distributions) have been presented for both the case of the two distributions following the same Poisson distribution and for the case of the two distributions following two different Poisson distributions. In both cases, the Poisson mean was 'perfectly-known', meaning that the actual mean is presumed known. (This case is actually a hypothetical one, since in reality a true Poisson mean is seldom, if ever, really known.) The calculational details were presented in the attached appendices. When the number of counts involved were sufficiently large, normal approximations were used (Appendix A); when not, the actual Poisson distributions were used (Appendices B and C). Selected illustrations of practical use were presented within the main text, both for the former case involving the same mean count and the latter case involving different mean counts.

These illustrations included (section 1) an analysis of expected check source response counts relative to an initial reference count, (section 2) an analysis of the Albatross (i.e., HPI model 2080B pulse neutron survey meter) neutron tube counts relative to the gamma tube counts within gamma fields, and (sections 3 and 4) an analysis of the use of the automatic background subtraction feature within COTS portable health physics instrumentation for both the beta and alpha channels, respectively.

In the first section involving normal approximations for the same mean count, the initial reference readings could have been established by an initial count characterized by the same counting time as used in the subsequent daily checks, or could have involved a count time that is ten times that used for each morning's daily check. When the calculations were extended to include reference count times ten times longer than the daily count time, the previously observed probabilities of exceeding the $\pm 10\%$ or $\pm 20\%$ criteria were significantly reduced. In other words, the variance in the daily difference was then dominated by the variance in that daily gross count rather than the variance in the initial determination of the reference value. These reduced probabilities in the tails were preferred, and indicated that when possible one should always choose a longer count time for the initial reference reading. The reader will have recognized the similarity to the reason behind determining background count rates using background count times that are ten times longer than the sample count times employed.

Additionally in the first section, the lowest nuisance rate of 0.04% was chosen such that 99.96% of a specific normal difference distribution stays within $\pm 10\%$. However, 98% of that distribution (corresponding to the 1-in-50 nuisance rate) stays within $\pm 6.57\%$. Therefore, an added benefit of having chosen the 0.04% rate is that bias (i.e., drifts) of up to a few percent can be somewhat tolerated. For the $\pm 20\%$ case, the associated 1-in-50 value is $\pm 13.14\%$, which is of course the same relative width, but which doubles the level of bias that could be tolerated (i.e., up to about 7%). This was an important reason why the level of reliability preferred throughout the paper was based on the negligible nuisance rate involving daily counts, i.e., 0.04%. The probability of a statistical outlier would therefore be expected to be quite small. If the result is outside the acceptable criterion, it probably isn't statistics, and an actual problem might need to be diagnosed.

None the less, it should also be noted that the minimum counts presented should not be seen as absolutes. Lower count values are acceptable, but the 'tail' probabilities will steadily increase with decreasing count values. If the vast portion of the distribution remains within the tolerance bands, however, then the consequence will be that now and then a couple of recounts will be necessary to confirm or deny a problem (presuming the use of a '2 out of 3' rule). Additionally, larger count values are also acceptable and recommended. It should be realized, however, that with significantly larger counts, the expected variations in response test readings will be significantly 'tighter' (i.e., with a relatively small statistical variance), and long-term drifts due to some real problem might actually go undetected for some time if closer attention (than simply ± 10 or 20%) is not given to the measurement data.

In the second section involving the actual Poisson distributions for the same mean count, it was seen that the Poisson difference distribution quickly converges to that from the normal approximation at Poisson means of approximately 10 counts. Of course, the Poisson mean was perfectly (or absolutely) known. As previously stated, this case is a hypothetical one, since in reality a true Poisson mean is seldom, if ever, really known. The Poisson difference distribution was also seen to be perfectly symmetric about zero as expected (seen in Fig. 7a, left side). The Poisson difference distributions were also found to be broader than the 'parent' Poisson distribution by the factor of $\sqrt{2}$, i.e., exactly that expected from classical statistics (Currie 1968). Equivalently, the variance in the net count (of zero) is doubled with respect to the parent since it is the sum of the background variance and the gross count variance (which is the same as the background variance for a net of 0 counts). Since the relative error in the Poisson difference distribution was still related to that of the parent Poisson distribution (i.e., $1/\sqrt{\mu}$), it was seen therefore in the analysis of the HPI2080B Albatross pulsed neutron remmeter that an effective way of reducing the gamma influence was to increase the counting interval from 16-s ($AV=1$) up to 128-s ($AV=8$) or more, hence yielding a significantly larger mean count per interval.

In the third and fourth sections, an analysis was performed of the use of the automatic background subtraction feature within COTS portable health physics instrumentation for both a

beta and an alpha channel, respectively. Each of the two measured values (i.e., the initial background count and the subsequent sample count) is randomly distributed about its own (and presumably different) Poisson mean count value. The quantity of interest is the difference of the two, which will be randomly distributed about the expected difference value, i.e., the mean difference of counts. Therefore, this application involved the expected random difference between two random distributions following two different Poisson distributions. This represents the possible measurement values with background subtraction activated within the instrument.

For either β or α counting channel of health physics instrumentation, i.e., as represented by the normal (Gaussian) based calculations or by the exact Poisson based calculations, respectively, the difference distributions were normalized and typically fairly symmetric about the mean difference value. Note that the difference distribution for the mean difference of 0 was perfectly symmetric.

It was seen that when the net count values consisted of relatively few counts, a significant portion of the associated background-subtract (i.e., difference) distribution consisted of members with negative values. This represents a serious problem with the COTS health physics instrumentation currently (and previously) supplied by major manufacturers, since all display a negative instrument reading as simple "0". An additional problem is that, at near background levels, the variance associated with the background subtracted readings will be essentially doubled over that of the gross (essentially) background measurement alone. It was also seen, however, that for net count values substantially greater than just a relatively few counts, the background subtracted reading will possess just slightly greater variance than if left as a gross mode measurement.

The gross mode of operation additionally allows the surveyor to actually observe the typically small range of background values in a given survey area. The effective background sampling time then involves observations of the background readings over relative long periods, such as the length of time to conduct a survey of some laboratory room (presumably several minutes). This isn't possible in the background subtraction mode of operation where one and only one background value is incorporated into the instrument. With some of the COTS health physics instrumentation, the exact value incorporated might not be known or readily available, at least in the same measurement unit as the displayed readings.

CONCLUSION

Certain health physics measurement applications involve the comparison of essentially two seemingly identical measurement results. One example of this particular application studied here was the comparison of a morning's scaler count of a reference source to an essentially identical count performed sometime previously that established the so-called reference reading. An additional example studied takes place within the so-called Albatross pulsed neutron remmeter, and involved the 'running' comparison of the Ag-wrapped neutron tube counts

relative to the Sn-wrapped gamma-compensation tube counts (both due to the same ambient gamma background). In the examples above, each of the two measurement results was randomly distributed about the same Poisson mean count value. The quantity of interest was the difference of the two results, which will be randomly distributed about the value of zero counts.

A slightly different application involved the comparison of two slightly different measurement results. The example of this particular measurement application studied here was the utilization of the automatic background subtraction feature within portable health physics instrumentation. Each of the two measurement results (i.e., the initial background count and the subsequent sample count) was randomly distributed about its own (and different) Poisson mean count value. The quantity of interest was the difference of the two results, which will be randomly distributed about the expected difference value, i.e., mean difference of counts.

This paper presented the calculational methods for the determination of both of these so-called difference distributions, i.e., both for the former case involving the same mean count and the latter involving different mean counts. At the lower count realm, a Poisson difference distribution was determined to be applicable. At the higher count realm, the normal difference distribution function was utilized. Specific examples were provided which highlighted the various concepts and illustrated the calculational methods.

With regard to the comparison of a daily source count to its previously-determined reference value, minimum counts for various scenarios were presented in order to reliably meet required tolerance limits of $\pm 10\%$ and $\pm 20\%$. In either case, it was found beneficial that the initial reference readings be established utilizing a counting interval of longer length than the daily interval. With regard to the automatic background subtraction feature, it was noted that net count values near 0 counts would almost always have the negative values of the difference distribution truncated to 0 counts by COTS instrumentation, whereas significant net count values would be displayed correctly, but with a larger associated variance than the gross count itself.

This paper therefore provides a technical basis for the necessary source strength of a check source in order to meet tolerance limits, the gamma field limitations of the HPI2080B Albatross, and the consequences of automatic background subtraction in COTS instrumentation.

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Appendix A. Calculations of the normal difference distribution between two independent variates following the same or different distributions.

The Normal Difference Distribution (Weisstein 2015) describes the distribution of a difference of two normally distributed variates X and Y with means and variances (μ_x, σ_x^2) and (μ_y, σ_y^2) , respectively. Its formula is given by eqn (A-1).

$$P_{X-Y}(d) = \frac{e^{-[d-(\mu_x-\mu_y)]^2/[2(\sigma_x^2+\sigma_y^2)]}}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}} \quad (\text{A-1})$$

where d is the difference value.

First, for the case involving the comparison of two measurements of the same mean count, $\mu_x = \mu_y$ and $\sigma_x^2 = \sigma_y^2 = \sigma^2$, i.e., same mean count and same count times, eqn (A-1) reduces to the formula given by eqn (A-2a).

$$P_{X-Y}(d) = \frac{e^{-d^2/4\sigma^2}}{\sigma\sqrt{4\pi}} \quad (\text{A-2a})$$

However, for $\mu_x = \mu_y$ and $\sigma_y^2 = 0.1 \times \sigma_x^2$, i.e., same mean count but with a reference count time ten times longer than the daily count time, eqn (A-1) reduces to the formula given by eqn (A-2b).

$$P_{X-Y}(d) = \frac{e^{-d^2/2.2\sigma_x^2}}{\sigma_x\sqrt{2.2\pi}} \quad (\text{A-2b})$$

Additionally, for $\mu_x = \mu_y$ and $\sigma_y^2 = 0.2 \times \sigma_x^2$, i.e., same mean count but with a reference count time five times longer than the daily count time, eqn (A-1) reduces to the formula given by eqn (A-2c).

$$P_{X-Y}(d) = \frac{e^{-d^2/2.4\sigma_x^2}}{\sigma_x\sqrt{2.4\pi}} \quad (\text{A-2c})$$

Note that the typical normal distribution is given by eqn (A-3),

$$P(x) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \quad (\text{A-3})$$

where x represents the possible values about the mean, μ . For a mean equal to zero, i.e., $\mu = 0$, and representing σ by instead the symbol ρ (the reason for which will soon become apparent), eqn (A-3) then reduces to the expression given by eqn (A-4):

$$P(x) = \frac{e^{-x^2/2\rho^2}}{\rho\sqrt{2\pi}} \quad (\text{A-4})$$

Note now that with the substitutions $\rho = \sigma\sqrt{2}$, $\rho = \sigma\sqrt{1.1}$, and $\rho = \sigma\sqrt{1.2}$, an equation of the form of eqn (A-3) can then represent eqn (A-2a), eqn (A-2b), and eqn (A-2c), respectively. This then allows the use of the built-in Excel function NORM.DIST(x , mean, std-dev, cumulative), where the mean is 0, the std-dev is ρ , and cumulative is either TRUE (i.e., 1) for the cdf or FALSE (i.e., 0) for the pdf. The differences are then determined by inspection of the resulting distribution. However, the built-in Excel function NORM.INV(prob, mean, std-dev) allows the direct determination of the difference value that yields the required one-sided cumulative probability (i.e., via the ‘prob’ parameter).

The equations above are directly applicable to alpha or neutron check source counts since the alpha or neutron background is essentially zero. However, this is not true for beta or gamma check source counts since beta and gamma backgrounds can be significant and need to be accounted for. Since a net count = the gross count – the background count, then the variance in the net, $\sigma_{\text{net}}^2 = \sigma_{\text{gross}}^2 + \sigma_{\text{bkgd}}^2 = \text{gross count} + \text{background count} = \text{net count} + 2 \times \text{background count} = \text{net count} \times (1 + 2 \times \text{background count/net count})$. Hence, the standard deviation in the net count, σ_{net} , is increased beyond the typical $\sqrt{(\text{net count})}$ as follows:

$$\sigma_{\text{net}} = \sqrt{\text{net}} \times \sqrt{1 + \frac{2 \times \text{bkgd}}{\text{net}}} \quad (\text{A-5})$$

where ‘net’ is the net mean count, and ‘bkgd’ is the background mean count.

Next, for the latter case involving the comparison of two slightly different mean counts, the value μ_y can be seen as the background mean count and the quantity $(\mu_x - \mu_y)$ as the net mean count, such that the value μ_x is seen as the gross mean count. Additionally, in this case, the variances are set equal to the applicable Poisson means. Hence, eqn (A-1) can be readily transformed to the more easily readable:

$$P_{\text{gross-bkgd}}(d) = P_{\text{net}}(d) = \frac{e^{-(d-\text{net})^2/(2(\text{gross}+\text{bkgd}))}}{\sqrt{2\pi(\text{gross}+\text{bkgd})}} \quad (\text{A-6})$$

where ‘bkgd’ is the background mean count, ‘net’ is the net mean count, and ‘gross’ is the gross mean count (i.e., equal to ‘bkgd’ + ‘net’).

Appendix B. Calculations of the Poisson difference distribution between two independent variates following the same Poisson distribution.

Several Excel-based calculations were performed, which represented the Poisson difference distribution between two measurements with the same Poisson mean count. The calculations were based on eqn (B-1) below. The calculations initially investigated Poisson mean counts per interval ranging from a 0.5 mean count up to 8 mean counts per counting interval. Some of the results were presented in the main text. Both the Poisson-based probability mass function (pmf) distributions were calculated as well as the cumulative distribution functions (cdf) for differences ranging from -16 counts to +16 counts for Poisson mean counts ranging from 0.5 to 8 mean counts per interval. As expected, all of the pmf distributions were both symmetric about zero and normalized. The increased broadening with increased mean count was quite apparent. It can be noted that the cdf's were not symmetric about zero, as is the case for the mathematical normal distribution, i.e., exactly half of the normal distribution is less than or equal to zero and exactly half is greater than or equal to zero. In the Poisson case, the same fraction of the distribution is less than zero as is greater than zero, but the largest probability for such a discrete distribution was for the difference of exactly zero. Therefore, exactly half of the distribution is not greater than or equal to zero, and that is reflected in the asymmetry of the cdf.

It was surmised that all the pmf distributions could be collapsed into a common curve. Indeed, by rebinning both axis by a factor equal to the original Poisson distribution's standard deviation, i.e., $\sqrt{(\text{mean count})}$, the difference distribution plots were found to coincide. It was also found that the width (or spread) of this difference distribution is greater than the original Poisson's by a factor of $\sqrt{2}$, as expected from the standard paired-blank case within classical statistics (Currie 1968).

As the calculations of the Poisson difference pmf distribution progressed, the expression of import evolved from eqn (B-1) to eqn (B-4) as presented below. The initial expression of import was the following product:

$$P_1(n_1, \mu) \cdot P_2(n_2, \mu) = \left(\frac{\mu^{n_1}}{n_1!} \cdot e^{-\mu} \right) \cdot \left(\frac{\mu^{n_2}}{n_2!} \cdot e^{-\mu} \right) = \frac{\mu^{n_1+n_2} \cdot e^{-2\mu}}{n_1! \cdot n_2!} \quad (\text{B-1})$$

where μ is the (identical) Poisson mean count expected for each of the two counts, n_1 are the possible integer count values possible from count #1, and n_2 are the possible integer count values possible from count #2. The probability of simultaneously observing a particular count n_1 in count #1 and n_2 in count #2 is simply $P_1 \times P_2$. By letting $n_2 - n_1 = d$, where d is the sought-after difference in the counts, the expression of import then became:

$$P(n_1, d, \mu) = \frac{\mu^{2n_1+d} \cdot e^{-2\mu}}{n_1! \cdot (n_1+d)!} \quad (\text{B-2})$$

The expression in eqn (B-2) represents the probability for a difference d , given a Poisson mean count μ and an observed count n_1 . This expression was entered into Excel and summed over a reasonably large number of rows and columns representing the possible n_1 values. At the low values of Poisson mean count considered, n_1 typically ranged from 0 to 40. This sum, however, should actually take place over the range of possible n_1 values from 0 to ∞ :

$$P(d, \mu) = \sum_{n_1=0}^{\infty} \frac{\mu^{2n_1+d} \cdot e^{-2\mu}}{n_1! \cdot (n_1+d)!} \quad (\text{B-3})$$

This expression was recognized, however, by Irwin (1937) to be equivalent to:

$$P(d, \mu) = e^{-2\mu} \cdot I_d(2\mu) \quad (\text{B-4})$$

where $I_d(2\mu)$ is the modified Bessel function of the first kind of order d and argument (2μ) . Note that this function is given by:

$$I_d(x) = \sum_{k=0}^{\infty} \frac{x^{d+2k}}{k! \cdot \Gamma(d+k+1) \cdot 2^{d+2k}} = \sum_{k=0}^{\infty} \frac{(x/2)^{d+2k}}{k! \cdot (d+k)!}, \text{ for } d \geq 0 \quad (\text{B-5})$$

The expression in eqn (B-4) represents the probability for a difference d simply given a Poisson mean count, μ . This expression was entered into Excel, using the built-in functions EXP and BESSELI, as simply $\text{EXP}(-2\mu) \times \text{BESSELI}(2\mu, d)$. This now allows the positive-sided Poisson difference pmf distributions to be readily determined, even for quite large values of the Poisson mean count μ , i.e., up to means of over 300 counts. The numeric results from the use of eqn. B-4 were identical to those from the initial (quite rigorous) use of eqn B-1. Note that $I_{-d}(x) = I_d(x) = I_{|d|}(x)$ readily allows the extension to $d < 0$. Alternately, symmetry about zero can be used to populate the negative values of the difference distribution, if needed.

Appendix C. Calculations of the Poisson difference distribution between two independent variates following different Poisson distributions.

Several Excel-based calculations were performed, which represented the Poisson difference distribution between two measurements with different Poisson mean counts. The calculations were initially based on eqn (C-1) below. The Poisson-based probability mass function (pmf) distributions were calculated for differences ranging from -10 counts to +30 counts for Poisson mean count differences ranging up to 10 counts. The increased broadening with increased mean count was again apparent. It should be noted that, with the exception of a difference of zero, the pmf distributions were not absolutely symmetric about the mean difference, although they were normalized.

As the calculations of the Poisson difference pmf distribution progressed, the expression of import evolved from eqn (C-1) to eqn (C-4) as presented below. The initial expression of import was the following product:

$$P_1(n_1, \mu_1) \cdot P_2(n_2, \mu_2) = \left(\frac{\mu_1^{n_1}}{n_1!} \cdot e^{-\mu_1} \right) \cdot \left(\frac{\mu_2^{n_2}}{n_2!} \cdot e^{-\mu_2} \right) = \frac{\mu_1^{n_1} \cdot \mu_2^{n_2} \cdot e^{-(\mu_1 + \mu_2)}}{n_1! \cdot n_2!} \quad (\text{C-1})$$

where μ_1 and μ_2 are Poisson mean counts expected for each of the two counts, n_1 are the possible integer count values possible from count #1 of μ_1 , and n_2 are the possible integer count values possible from count #2 of μ_2 . The probability of simultaneously observing a particular count n_1 in count #1 and n_2 in count #2 is simply $P_1 \times P_2$. By letting $n_2 - n_1 = d$, where d is the sought-after difference in the two counts, the expression of interest then became:

$$P(n_1, d, \mu_1, \mu_2) = \frac{\mu_1^{n_1} \cdot \mu_2^{n_1+d} \cdot e^{-(\mu_1 + \mu_2)}}{n_1! \cdot (n_1+d)!} \quad (\text{C-2})$$

The expression in eqn (C-2) represents the probability for a difference d , given Poisson mean counts of μ_1 and μ_2 and an observed count n_1 . This expression was entered into Excel and summed over a reasonably large number of rows representing the possible n_1 values and columns representing n_2 values. For low values of Poisson mean counts, n_1 typically ranged from 0 to 50. This sum, however, should actually take place over the range of possible n_1 values from 0 to ∞ :

$$P(d, \mu_1, \mu_2) = \sum_{n_1=0}^{\infty} \frac{\mu_1^{n_1} \cdot \mu_2^{n_1+d} \cdot e^{-(\mu_1 + \mu_2)}}{n_1! \cdot (n_1+d)!} \quad (\text{C-3a})$$

$$= \sum_{n_1=0}^{\infty} \frac{\mu_2^{\frac{d}{2}} \cdot \mu_1^{\frac{1}{2}(2n_1+d)} \cdot \mu_2^{\frac{1}{2}(2n_1+d)} \cdot e^{-(\mu_1 + \mu_2)}}{\mu_1^{\frac{d}{2}} \cdot n_1! \cdot (n_1+d)!} \quad (\text{C-3b})$$

This expression can now be recognized to be equivalent to:

$$P(d, \mu_1, \mu_2) = e^{-(\mu_1 + \mu_2)} \cdot \left(\frac{\mu_2}{\mu_1} \right)^{\frac{d}{2}} \cdot I_{|d|}(2\sqrt{\mu_1 \mu_2}) \quad , \text{ for } \mu_2 \geq \mu_1 \quad (\text{C-4})$$

where $I_d(x)$ is the modified Bessel function of the first kind of order d and argument (x) , given previously in eqn (B-5), μ_2 must be greater than or equal to μ_1 , and that $I_{-d}(x) = I_d(x) = I_{|d|}(x)$. This was first presented in problem #9 of section V of Feller (1966), but without the important condition that $\mu_2 \geq \mu_1$.

The expression in eqn (C-4) represents the probability for a difference d given the two Poisson mean counts. This expression was entered into Excel, using the built-in functions EXP, SQRT, and BESSELI. This now allows the Poisson difference pmf distribution to be readily determined, even for quite large values of Poisson mean count, i.e., up to means of over 300 counts. The numeric results from the use of eqn. C-4 were identical to those from the initial (quite rigorous) use of eqn C-1. Note that unlike the case in Appendix B involving the same Poisson mean, perfect symmetry does not occur about the expected difference value, although they were normalized. Note also that when $\mu_2 = \mu_1 = \mu$, eqn (C-4) reduces to eqn (B-4).